Auctions with Intermediaries

[Extended Abstract]

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ABSTRACT

Inspired by online advertisement exchange systems, we study a setting where potential buyers of a unique, indivisible good attempt to purchase from a *central seller* via a set of *intermediaries*. Each intermediary has captive buyers, and runs an auction for a 'contingent' good. Based on the outcome, the intermediary bids in a subsequent upstream auction run by the seller. In this paper, we study the equilibria and incentives of intermediaries and the central seller.

We find that combining the notion of optimal auction design with the double-marginalization arising from the presence of intermediaries yields new strategic elements not present in either setting individually: we show that in equilibrium, revenue-maximizing intermediaries will use an auction with a *randomized reserve price* chosen from an *interval*. We characterize the interval and the probability distribution from which this reserve price is chosen as a function of the distribution of buyers' types. Furthermore, we characterize the revenue maximizing auction for the central seller by taking into account the effect of his choice of mechanism on the mechanisms offered by the intermediaries. We find that the optimal reserve price offered by the seller decreases with the number of buyers (but remains strictly positive); this is in contrast to classical auctions without intermediaries, where the optimal reserve price is independent of the number of buyers.

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1. INTRODUCTION

In a classical auction setting, we have a seller with a good, and interested buyers place their bids with the seller. The winner and the price she pays are determined by the auction. A rich theory of auction design has been developed over the past few decades to reason about this setting (see, e.g. [7]). In contrast to this, our focus here is on auction settings where there are intermediaries: the buyer, rather than buying from the seller directly, chooses (or is required) to buy via intermediaries.

Our motivation arises from ad auctions on the Internet. In sponsored search auctions, online advertisers may use intermediaries to bid on behalf of their campaigns, and indeed there is a large industry of such intermediaries that attempt to optimize campaign returns by careful bidding. Likewise, in the display ad business, advertisers have long relied on ad agencies as intermediaries to aggregate across a wide variety of publishers and secure premier ad slots. Increasingly, display ads are sold via exchanges where ad networks act as intermediaries and bid on behalf of advertisers in real-time auctions.

In this paper we study optimal auction design in the presence of intermediaries. Our perspective is Bayesian in that intermediaries and the main seller have probabilistic information about the private valuation of buyers. We find that the presence of intermediaries produces fundamental new elements of bidding strategy not present in a classical auction; for example we show that intermediaries use *non-deterministic* reserve prices in equilibrium. Further, the seller's optimal reserve price changes with the number of bidders, in contrast to the standard buyers and monopolist seller case (see Myerson [12]).

1.1 Auctions with Intermediaries: Our Setting

Here we give a general outline of the setting we study and its motivation; a formal description is in Section 2. There is a unique, indivisible good to sell at the *center*. There are n *intermediaries* that bid in an auction run by the center. The center and all intermediaries have no value for the good themselves. Each intermediary i has a set of buyers for whom they are the exclusive agent. As is standard in the Independent Private Values (IPV) setting, each buyer j

has a value for the good v_j which is assumed to be an i.i.d. draw from some distribution f over $[0,1]$. Throughout the paper, we make a standard assumption about f satisfying the monotone hazard-rate condition. While the value realization v_j is his private information, the distribution f is common knowledge.

The timing of the auction process is as follows: first each intermediary runs (independently in parallel) a *contingent auction*. This auction determines a contingent buyer and a contingent price for that intermediary, i.e. the identity of which buyer will get the good *if* the intermediary gets the good, and the price she will pay the intermediary. Then the center runs an auction for the good among intermediaries. The intermediaries bid in the center's auction based on the outcome in their own contingent auctions. The winning intermediary then transfers the good as determined by its auction.

This formulation is general. Our motivation arose from online display ads. In particular, an emerging trend is that display ads are transacted on real-time marketplaces such as RightMedia,[1] AdECN[2] and Doubleclick Ad Exchange.[3] These marketplaces were abstracted into the AdX Model in [11] where each impression of ads is auctioned in real-time. In the AdX model [11], when the exchange becomes aware of the good to sell (an ad slot impression), it contacts the ad networks who return a bid for an impression of their customer's ad. Thus the exchange in the AdX Model is the center in our setting, the ad networks are the intermediaries, and the advertisers are the buyers. The formulation also fits other instances of intermediaries such as optimized bidding for sponsored search advertisers, or bidding on art, real estate or T-bills and other goods.

A standard concern in economics that relates to our setting is that of *double marginalization* [8, 15, 3]. An upstream monopoly manufacturer sells to customers via a monopolist downstream retailer. In contrast to the case where the manufacturer sells directly to the customer, the consumer will be charged a higher price and therefore demand a lower quantity, since both the manufacturer and the retailer must extract margins from this transaction. Our model studies double marginalization in the context of auctions, where the standard literature studies this where the manufacturer and retailer select prices.

1.2 Our Contribution

We study the incentives of, and equilibrium among, buyers, intermediaries and the central seller in a Bayesian setting. We show that even in the case of one buyer per intermediary— which at first glance seems like a straightforward application of Myerson's results– the equilibrium strategies, and auction designs are markedly different.

- *[Intermediaries]* We study the incentives of intermediaries in designing their downstream auctions as a function of the upstream auction announced by the center, and characterize the equilibrium mechanisms offered in a game-theoretic setting.

  We show that the revenue-maximizing strategy of intermediaries in an equilibrium is to apply a *randomized reserve price* r chosen from an *interval* $[\underline{v}_r, \bar{v}_r]$, and determine the interval $[\underline{v}_r, \bar{v}_r]$ and the probability distribution π based on which the reserve price r is chosen (Theorem 2). Using this analytical characterization, we show both analytical and numerical properties of such equilibria. In particular, we observe that $\underline{v}_r$ is independent of the number of intermediaries, $\bar{v}_r$ increases as the number of intermediaries increases (Theorem 3), and the mass of distribution π moves toward $\bar{v}_r$ as the number of intermediaries increases.

  The intuition for our qualitative results is easily explained. In our setting, the profit of an intermediary is the difference between the amount buyers pay them and the amount they owe the center *when* they win the good in the center's auction; and nothing otherwise. The randomization allows them to be unpredictable in the eyes of other intermediaries. If, by contrast, they used a fixed reserve price, another intermediary could announce a reserve price just slightly higher and exploit. Similarly, as the number of intermediaries grows large, each intermediary has a diminishing chance of having the highest valued buyer. The intermediary compensates by selecting an ever higher reserve price, to increase both its chance of winning, and the contingent price paid by the buyer conditional on the reserve price being met.

  This is different from the traditional optimal auction setting without intermediaries [12] in which the reserve price is a fixed number and is independent of the number of buyers. In a simple double marginalization setting without a surrounding auction, the price set by the intermediary is a particular function of the buyer valuation distribution; in contrast, in our setting the mediators *must necessarily randomize their prices* in equilibrium.

- *[Center]* We study the revenue maximizing auction for the seller taking into account the effect of his/her choice of mechanism on the mechanisms offered by the intermediaries.

  We characterize center's optimal auction and present analytical and numerical results for the optimal reserve price used by the seller.

  We show that the optimal price decreases as the number of intermediaries increases(Figure 4), however the reserve price remains strictly positive (Theorem 5). Again this is in contrast with the traditional auction setting without intermediaries, where the optimal reserve is independent of the number of participants. [12].

Finally, we show how to extend these results to the case with many buyers per intermediary. Qualitatively, the observations above hold, but the mathematical analyses become more involved since multiple reserve prices and equilibrium distributions are needed for randomized bidding.

We get our results above by considering the two different, dependent optimization problems of revenue maximization at the intermediaries and the center. At the center, the problem P1 optimizes the revenue of the center conditioned on the bids of the intermediaries. At the intermediaries, the problem P2 optimizes the revenue conditioned on bidding strategies of other intermediaries. Each of these problems looks like the traditional revenue optimization problem P

[1] http://www.rightmedia.com
[2] http://www.adecn.com
[3] http://www.doubleclick.com

without intermediaries where there is a seller and multiple bidders. However, P2 differs from P in that the good is only contingent, and therefore, the probability of winning the good at the center is critical and this depends on the other intermediaries and their strategies; likewise, P1 differs from P since the bidders choose their mechanisms in response to the outcome of center's optimization unlike in P.

Further, P1 and P2 must be solved simultaneously in equilibrium since each affects the other. We get our technical results above by studying both simultaneously, but using a series of structural observations about the equilibria of optimal auctions to identify the right parameterized versions of these problems to be studied separately. (For example, we study the intermediary's problem parameterized by the choice r of reserve price by the center and $\alpha(b)$, the probability of an intermediary winning the good with bid b at the center.) We believe these structural observations are of interest by themselves and should be useful for studying the equilibrium of optimal auctions in other setting with intermediaries.

Our setting abstracts two levels of auctions in the presence of intermediaries. This understanding is of fundamental value in designing and reasoning about modern ad exchanges [11] because seen from the vantage of the exchange, this view is of most immediate interest.

As exchanges look beyond, one may be interested in models that feature multiple levels of intermediaries, each with multiple customers; further, customers who are common to multiple intermediaries, each intermediary may access multiple exchanges, there are side contracts between intermediaries and between exchanges, and so on. It would be interesting future research to develop a theory of optimal mechanisms in presence of such an interconnected network of intermediaries and buyers, however such generalizations seem to be very technical [13]. We conclude the paper discussing this and other open research directions.

1.3 Related Work

Auction design in the presence of competing auctioneers who are competing over a set of advertisers have been recently studied in the literature [1, 13], however, this work does not consider an extra layer of a seller selling an item to these competing auctioneers. In this line of work, the dynamics of buyers changing their choice of which auctioneer to buy from plays an important role, and in fact, makes the auction design problem very technical [13]. In this paper, we do not model the dynamics of buyers switching among auctioneers, and focus on competing auctioneers each with their own customers, but competing to buy a contingent good from a central seller.

Designing efficient and revenue-maximizing truthful mechanisms is a central issue in the mechanism design and has been studied extensively[16, 5, 12]. A standard way for maximizing revenue is to derive some value profile from the bids, calculate bidder-specific reserve price, and runs a second-price auction [12, 2, 14]. Prior-free revenue-maximizing mechanisms have been developed for various auction settings [4, 6]. Lower bounds show that prior-free truthful auction cannot achieve revenue comparable to the revenue-optimal auctions with prior [6, 9].

Ad clearinghouses are similar to bidding rings studied in McAfee and Mcmillan [10], except that here the bidding ring center (in the terminology of [10]) is an agent looking to maximize its own profit, while in [10] it was simply a co-ordinating device for the bidders in the ring.

2. OUR SETTING

There is a central auction house called the *center* with one unique, indivisible good to sell. There are n *intermediaries* attempting to buy a good from the central auction house. The center and intermediaries have no value for the good itself. Each intermediary i has j_i buyers present, and contracted with it. The number of buyers at each intermediary is a common knowledge. Each buyer has a value for the good v which is an i.i.d. draw from some distribution f, F over $[0, 1]$. This valuation is private information, but the distribution f, F is common knowledge. We will maintain throughout that f, F satisfy the *monotone hazard rate assumption*, i.e., $\forall v \in [0, 1]: \quad \frac{f(v)}{1-F(v)} \uparrow$ in v; this assumption is standard in mechanism design since [12].

The timing of the game is as follows:

1. The buyers learn their private valuations for the good.
2. The center announces what auction it is going to run.
3. Each of the intermediaries runs an auction to sell a *contingent* good. In other words, it sells to its advertisers the obligation to buy the good at a specific price if it buys the good in the upstream auction.
4. The center now runs the announced auction to sell the good. Transfers good to winner (if any), collects payment.
5. The winning intermediary (if any) transfers the good to the contingent winner in its own auction, and collects payment.

Note that the center will look to maximize its expected revenue, while intermediaries will look to maximize their expected profits (i.e. the amount they collect from buyers less the amount they need to pay out to the center).

We posit that the auction run by the center is a second-price auction with a suitable reserve r. First, from a mathematical point of view, when the center's mechanism is run, each of the intermediaries has already run an auction for the contingent good, and therefore each intermediary knows his value for the good: the price agreed upon with the winning buyer. Thus seen from the vantage of the center, it is simply running an auction for a single indivisible item with bidders with known values. From Myerson [12] we know that the revenue-optimal auction in such a case is a second-price auction with reserve. This is a good justification for us, even though there are caveats: the second-price auction is only revenue-optimal in the case of symmetric participants, which in this setting means that each intermediary has the same distribution on the bid it will submit to the auction. Such symmetry could result from the intermediaries having the identical number of buyers, and those buyers having the identical distribution. While this assumption is not true in our motivating applications such as online ad exchanges, it is still reasonable to assume that the second-price auction is an attractive mechanism for an open ad exchange. This is because ad exchanges desire a symmetric mechanism in order to be open platforms that are fair toward different intermediaries. Further, second-price auctions are the predominantly used mechanisms in online ad auctions in practice. In particular, the DoubleClick Ad Exchange uses second-price auction.[4]

Given this formulation, the main questions are as follows:

1. How will intermediaries design their auction given their incentives and choice of auction used by the center? In particular, we will be able to prove that in the case of a single buyer per intermediary i, the intermediary must offer a single posted price R_i. How will R_i's depend on center's choice?

2. How will a rational center design its auction given that its forecast of auctions intermediaries will choose? In particular, how will r vary?

3. How is the ensuing equilibrium distorted away from the case when buyers directly bid with the center? In particular, what is the cost of intermediaries in the equilibrium?

These are fundamental questions that we address in this paper.

We will use $\alpha(b)$ to denote the probability that a particular intermediary will win the contingent good, given that they bid b in the center's auction. Because the center runs a second-price auction, note that $\alpha(\cdot)$ is monotone.

3. INTERMEDIARY

We consider the intermediaries' problem, and we will require a dominant strategy incentive-compatible symmetric mechanism, with ex-post individual rationality. By an application of the revelation principle we can restrict attention to direct revelation mechanisms: each intermediary requires the buyers to report their valuations and commits to an allocation and payment rule as a function of the reported values. Let $a(v)$ be the probability of winning the contingent good (the obligation to buy the good) if a buyer bids v, and $p(v)$ be the contingent payment from the buyer to the intermediary (contingent on the intermediary winning the good upstream).

We will use the following proposition; its proof is standard but included in the Appendix for completeness.

PROPOSITION 1. *[12] An auction is revenue-optimal, incentive compatible and individually rational iff its allocation function $a(\cdot)$ is increasing and the price function $p(\cdot)$ satisfies*

$$p(v) = va(v) - \int_0^v a(x)dx.$$

3.1 Intermediary's Choice of Mechanism

If the buyer agrees to pay the intermediary a price p, and the intermediary bids p into the center's auction (since bidding truthfully is a dominant strategy), the intermediary's expected profit is $\int_0^p \alpha(v)dv$ using Proposition 1. As a result, under a strong incentive compatibility condition (IC) and individual rationality (IR), the a, p that maximizes the intermediary's profits (given α) is a solution to the following optimization problem:

$$\begin{aligned}
&\max_{a,p} \ \mathbb{E}_v\left(\int_0^{p(v)} \alpha(b)db\right), && \text{(IntOPT)}\\
&\text{s.t. } \forall v: && \\
&\quad \forall \text{ monotone } \alpha', && \text{(IC)}\\
&\quad \alpha'(p(v))\big(va(v)-p(v)\big) \geq \alpha'(p(v'))(va(v')-p(v')) && \\
&\quad va(v)-p(v) \geq 0 && \text{(IR)}\\
&\quad 0 \leq a(v) \leq 1 && \text{(Feasibility)}
\end{aligned} \qquad (1)$$

Note a main difference between this optimization problem and a "standard" auction design problem is in the IC constraints, where an $\alpha(\cdot)$ appears. This encapsulates incentive compatibility in this setting, since the intermediaries are not actually selling the good, they are selling the obligation to buy the good at a specific price p, and this price will affect the probability of their actually winning the good in the center's auction. This is since this price will be the intermediary's bid in the center's second-price auction. The following example illustrates this point:

Example. Suppose for this example that for a particular intermediary, we have $\alpha(b) = b$ for $b \in [0,1]$. Suppose this intermediary offers a single buyer with valuation $v = \frac{3}{4}$ two options: either pay $p_1 = \frac{1}{2}$, or pay $p_2 = \frac{1}{8}$. Under a simple auction, the buyer will clearly prefer p_2 for a profit of $v - p_2 = \frac{5}{8}$ (rather than $v - p_1 = \frac{1}{4}$). However in the contingent auction, the buyer will actually prefer to pay p_1, since it's expected profit is $\alpha(p_1)(v-p_1) = \frac{1}{8}$, rather than $\alpha(p_2)(v - p_2) = \frac{5}{64}$. □

Despite this phenomenon, it turns out that offering such a "menu" of prices cannot result in a mechanism for which truth-telling is actually a dominant strategy under the strong notion of incentive compatibility in problem (IntOPT). This is because in such a mechanism, bidding v must dominate all other bids, regardless of what other buyers do (in other intermediaries' auctions), *and* regardless of the behavior of other intermediaries (in terms of the auction they run, and how they bid in the center's auction). We prove this claim:

THEOREM 1. *With a single buyer, a take-it-or-leave-it price is the only dominant-strategy incentive-compatible mechanism available to the intermediary.*

PROOF. A take-it-or-leave-it price mechanism has the following form: $p(v) = p$, $a(v) = 1$ for $v \geq t$, and $a(v) = 0$ for $v < t$. Suppose for the sake of contradiction that the mechanism is not a take-it-or-leave-it price. Then, either $a(v)$ does not have this simple form, or $p(v)$ is not constant for $v \geq t$. However Proposition 1 (with $\alpha'(v) = 1$ for all v) rules out the latter case, since (1) gives $p(v)$ as a function only of $a(v)$. Now, by monotonicity of $a(v)$ (also from Proposition 1), we may assume that $a(v)$ does not have this form, i.e., there exists $v_1 < v_2$ such that $0 < a(v) < 1$ for $v \in [v_1, v_2]$.[5]

Pick some v' such that $v_1 < v' < v_2$, and let $\epsilon > 0$ be such that $\epsilon < \frac{(v'-v_1)a(v_1)}{2a(v_2)}$. Let $\alpha'(v)$ be a monotone function such that $\alpha'(p(v')) = 0$ and $\alpha'(p(v'+\epsilon)) = 1$. Using the incentive compatibility constraint for this $\alpha'(v)$ we get:

[4] http://adwords.google.com/support/aw/bin/answer.py?hl=en&answer=146606

[5] If $a(v) \notin \{0,1\}$ for exactly one point v_1, then the resulting mechanism is not distinguishable from a take-it-or-leave it price of v_1. This is because the only distinction occurs when the buyer has value exactly v_1. Since the distribution of buyer values has a density, this occurs with probability 0.

$$
\begin{aligned}
0 =&\alpha'(p(v'))(v'a(v') - p(v')) \\
\geq&\alpha'(p(v'+\epsilon))(v'a(v'+\epsilon) - p(v'+\epsilon)) \\
=&v'a(v'+\epsilon) - p(v'+\epsilon) \\
=&v'a(v'+\epsilon) - ((v'+\epsilon)a(v'+\epsilon) - \int_0^{v'+\epsilon} a(x)dx) \\
=&-\epsilon a(v'+\epsilon) + \int_0^{v'+\epsilon} a(x)dx \\
\geq&-\epsilon a(v'+\epsilon) + (v'+\epsilon - v_1)a(v_1) \\
\geq&-\epsilon a(v'+\epsilon) + (v' - v_1)a(v_1) \\
\geq&-\frac{(v'-v_1)a(v_1)}{2a(v_2)}a(v'+\epsilon) + (v'-v_1)a(v_1) \\
=&(v'-v_1)a(v_1)(1 - \frac{a(v'+\epsilon)}{2a(v_2)}) > 0
\end{aligned}
$$

deriving the contradiction. □

For the remainder of this section, we will consider the case of one buyer per intermediary; qualitatively, the extension to the general case will be in Section 5.

3.2 Intermediary's Mechanism: Equilibrium

Suppose an intermediary offers the price p to the buyer. In the take-it-or-leave-it mechanism, buyers with valuations $v \geq p$ will agree to this price, buyers with valuations less than p will reject. If the buyer accepts p, the intermediary will then bid p into the center's auction. Therefore the intermediary's expected profit given he offers a price p to the buyer is: $\text{Profit}(p) = (1-F(p))\int_0^p \alpha(b)db$. As a result the price that maximizes profit for the intermediary satisfies (assuming interior solution)

$$\frac{1-F(p^\star)}{f(p^\star)} = \frac{\int_0^{p^\star}\alpha(p)dp}{\alpha(p^\star)}. \qquad \text{(FOC)}$$

Consider the symmetric equilibrium among intermediaries. Observe that the equilibrium cannot involve intermediaries using pure strategies. Suppose all the other sellers were offering their buyer a price p. Then seller 1 is better off offering a price of $p+\epsilon$ for some small epsilon. For analogous reasons there cannot be atoms in any mixed strategy equilibrium. In fact,

LEMMA 1. *Any symmetric mixed equilibrium leaves the intermediary indifferent over exactly an interval of prices* $[\underline{v}_r, \bar{v}_r]$.

PROOF. Suppose not. Suppose the support of equilibrium prices offered by the mechanism contains $(..., v_1]\bigcup[v_2, ...)$ and does not contain (v_1, v_2). Therefore for any $v \in (v_1, v_2)$, $\alpha(v) = \alpha(v_2)$: since no other network offers prices in (v_1, v_2). Therefore, for any price $p \in (v_1, v_2]$, the seller's revenue is:

$$
\begin{aligned}
\text{Profit}(p) &= (1-F(p))\int_0^p \alpha(b)db \\
&= (1-F(p))\Big(\int_0^{v_1}\alpha(b)db + (p-v_1)\alpha(v_2)\Big) \\
&= (1-F(p))(p\alpha(v_2) + c).
\end{aligned}
$$

where $c = \int_0^{v_1}(\alpha(v) - \alpha(v_2))dv$. We know that for this to be an equilibrium, $\text{Profit}(p) < \text{Profit }(v_i)$, $i = 1, 2$, and Profit $(v_1) = \text{Profit }(v_2)$. However, differentiating Profit wrt p,

$$
\begin{aligned}
&= (1-F(p))\alpha(v_2) - f(p)(c + \alpha(v_2)p). \\
&= \alpha(v_2)\Big(1 - F(p) - pf(p)\Big) - cf(p). \\
&= -\Big(p - \frac{1-F(p)}{f(p)}\Big)f(p)\alpha(v_2) + pf(p)\alpha(v_2) - f(p)\int_0^{v_1}(\alpha(v))dv. \\
&= f(p)\Big(-\int_0^{v_1}(\alpha(v))dv + \frac{1-F(p)}{f(p)}\alpha(v_2)\Big) \\
&= f(p)\alpha(v_2)\Big(\frac{-\int_0^{v_1}(\alpha(v))dv}{\alpha(v_2)} + \frac{1-F(p)}{f(p)}\Big)
\end{aligned}
$$

But this must equal 0. To see this, note that $\text{Profit}(p) < \text{Profit }(v_1)$ for $p \downarrow v_1$

$$\Rightarrow \frac{-\int_0^{v_1}(\alpha(v))dv}{\alpha(v_2)} + \frac{1-F(v_1)}{f(v_1)} < 0$$

But by monotone hazard rate assumption, for all $p \in (v_1, v_2)$

$$\frac{-\int_0^{v_1}(\alpha(v))dv}{\alpha(v_2)} + \frac{1-F(p)}{f(p)} < 0,$$

which would imply that $\text{Profit}(v_1) > \text{Profit}(v_2)$. □

Next we state and prove the main theorem of this section:

THEOREM 2. *There exists a symmetric equilibrium in mixed strategies among the intermediaries, where each intermediary offers prices* $v \in [\underline{v}_r, \bar{v}_r]$ *with density* $\pi_r(v)$, *where* $\underline{v}_r, \bar{v}_r$ *and* $\pi(\cdot)$ *jointly solve:*

$$\underline{v}_r = r + \frac{1-F(\underline{v}_r)}{f(\underline{v}_r)} \qquad (2)$$

$$\int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)}{(n-1)\theta^{\frac{n-2}{n-1}}(v)(1-F(v))}dv = \big(\theta(\bar{v}_r)\big)^{\frac{1}{n-1}} \qquad (3)$$

$$\pi_r(v) = \left(\frac{1}{\theta(\bar{v}_r)}\right)^{\frac{1}{n-1}} \frac{\theta'(v)}{(n-1)\theta^{\frac{n-2}{n-1}}(v)(1-F(v))} \qquad (4)$$

where $\theta(v) \equiv \frac{f(v)}{(1-F(v))^2}$.

PROOF. For (2, note that for all $v \in (r, \underline{v}_r)$, $\alpha(v) = \alpha(\underline{v}_r)$. Therefore,

$$\text{Profit}(v) = (1-F(v))(v-r)\alpha(\underline{v}_r).$$

For this to be an equilibrium, we require that $\text{Profit}(v) < \text{Profit}(\underline{v}_r)$, i.e. that $(1-F(v))(v-r) < (1-F(\underline{v}_r))(\underline{v}_r - r)$. However, the derivative of the function $(1-F(v))(v-r)$ w.r.t. v is $f(v)\Big(\frac{1-F(v)}{f(v)} - (v-r)\Big)$. $v - r$ is increasing in v, while $\frac{1-F(v)}{f(v)}$ is decreasing in v by monotone hazard rate assumption. Therefore $(1-F(v))(v-r) < (1-F(\underline{v}_r))(\underline{v}_r - r)$. requires that $\underline{v}_r$ be s.t. $v - \frac{1-F(v)}{f(v)} \leq r$. Next, substituting $\int_0^{\underline{v}_r}\alpha(v)dv = (\underline{v}_r - r)\alpha(\underline{v}_r)$ into FOC, $\underline{v}_r$ must solve $v - \frac{1-F(v)}{f(v)} = r$.

For the networks to be indifferent over the interval $[\underline{v}_r, \bar{v}_r]$, we require that $\alpha(\cdot)$ solve (FOC) along the entire interval $[\underline{v}_r, \bar{v}_r]$. This implies that:

$$\alpha_r(v) = c_r\frac{f(v)}{(1-F(v))^2}. \qquad (5)$$

To pin down c_r and $\bar{v}_r$, we have the two following equations. First, clearly $\alpha_r(\bar{v}_r) = 1$. Second, let the randomization used by each seller over $[\underline{v}_r, \bar{v}_r]$ be π_r. Therefore:

$$\alpha_r(v) = \Big(\int_{\underline{v}_r}^{v} \pi_r(t)dt + \int_{v}^{\bar{v}_r} \pi_r(t)F(t) \Big)^{n-1}.$$

This implies that:

$$\alpha_r'(v) = (n-1)\alpha_r(v)^{\frac{n-2}{n-1}}(1-F(v))\pi_r(v),$$

for $v \in [\underline{v}_r, \bar{v}_r]$, i.e.:

$$\pi_r(v) = \frac{1}{n-1} \frac{\alpha_r'(v)}{\alpha_r(v)^{\frac{n-2}{n-1}}(1-F(v))}. \quad v \in [\underline{v}_r, \bar{v}_r] \qquad (6)$$

Substituting in the definition of α_r, (4) follows.

Since π_r must be a probability density function: $\int_{\underline{v}_r}^{\bar{v}_r} \pi_r(v) = 1$. In other words, $\bar{v}_r$ and c_r must jointly solve: $c_r \frac{f(\bar{v}_r)}{(1-F(\bar{v}_r))^2} = 1$ and $\int_{\underline{v}_r}^{\bar{v}_r} \pi_r(v) = 1$. Substituting definition of θ, we have $c_r = \frac{1}{\theta(\bar{v}_r)}$. which in turns gives (3). □

THEOREM 3. *Suppose the interval $[\underline{v}_r, \bar{v}_r]$ and the probability distribution $\pi_r(\cdot)$ solve Equations (2- 4). Then*

1. *Parameters $\underline{v}_r, \bar{v}_r$ are increasing in the center's reserve price r holding the number of intermediaries n fixed.*
2. *The parameter $\bar{v}_r$ is increasing in the number of intermediaries n holding center's reserve price r fixed.*

PROOF. To see the former, note recall that $\underline{v}_r$ solves

$$v - \frac{1-F(v)}{f(v)} = r,$$

and the left hand side is increasing in v by the monotone hazard rate assumption.

To see that $\bar{v}_r$ is increasing in r, note that:

$$\int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)}{(n-1)\theta^{\frac{n-2}{n-1}}(v)(1-F(v))} dv = \big(\theta(\bar{v}_r)\big)^{\frac{1}{n-1}}$$

$$\Rightarrow \frac{\theta'(\bar{v}_r)}{\theta^{\frac{n-2}{n-1}}(\bar{v}_r)} \frac{d\bar{v}_r}{dr} \Big(\frac{1}{1-F(\bar{v}_r)} - 1 \Big) = \frac{\theta'(\underline{v}_r)}{\theta^{\frac{n-2}{n-1}}(\underline{v}_r)(1-F(\underline{v}_r))} \frac{d\underline{v}_r}{dr} \qquad (7)$$

where the latter equality follows from differentiating both sides of the equation by r.

The result follows since $\theta'(\cdot) > 0$ and $\frac{d\underline{v}_r}{dr} \geq 0$.

To see the latter, note that $\bar{v}_r$ solves:

$$\int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)}{(n-1)\theta^{\frac{n-2}{n-1}}(v)(1-F(v))} dv = \big(\theta(\bar{v}_r)\big)^{\frac{1}{n-1}}$$

Define $x = \frac{1}{n-1}$.

$$\Rightarrow \int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)}{\theta^{1-x}(v)(1-F(v))} dv = \frac{\theta^x(\bar{v}_r)}{x}. \qquad (8)$$

Differentiating with respect to x:

$$\frac{\theta'(\bar{v}_r)}{\theta^{1-x}(\bar{v}_r)(1-F(\bar{v}_r))} \frac{d\bar{v}_r}{dx} + \int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)\ln\theta(v)}{\theta^{1-x}(v)(1-F(v))} dv$$
$$= \frac{\theta'(\bar{v}_r)}{\theta^{1-x}(\bar{v}_r)} \frac{d\bar{v}_r}{dx} - \frac{\theta^x(\bar{v}_r)}{x^2} + \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)}{x}$$
$$\Rightarrow \frac{\theta'(\bar{v}_r)F(\bar{v}_r)}{\theta^{1-x}(\bar{v}_r)(1-F(\bar{v}_r))} \frac{d\bar{v}_r}{dx} = -\frac{\theta^x(\bar{v}_r)}{x^2} + \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)}{x}$$
$$- \int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)\ln\theta(v)}{\theta^{1-x}(v)(1-F(v))} dv$$

Therefore if we show the right hand side is negative, we are done, since clearly $\frac{d\bar{v}_r}{dx} < 0 \Rightarrow \frac{d\bar{v}_r}{dn} > 0$.

Therefore we need to show that :

$$-\frac{\theta^x(\bar{v}_r)}{x^2} + \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)}{x} - \int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)\ln\theta(v)}{\theta^{1-x}(v)(1-F(v))} dv \leq 0$$
$$\iff -\frac{\theta^x(\bar{v}_r)}{x^2} + \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)}{x}$$
$$- \frac{1}{x}\int_{\underline{v}_r}^{\bar{v}_r} \frac{1}{(1-F(v))} \Big(\frac{d}{dv}(\theta^x(v)\ln\theta(v)) - \frac{\theta'(v)}{\theta^{1-x}(v)} \Big) dv \leq 0$$
$$\iff \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)}{x}$$
$$- \frac{1}{x}\int_{\underline{v}_r}^{\bar{v}_r} \frac{1}{(1-F(v))} \Big(\frac{d}{dv}(\theta^x(v)\ln\theta(v)) \Big) dv \leq 0 \qquad \text{(by (8))}$$
$$\iff \frac{1}{x}\int_{\underline{v}_r}^{\bar{v}_r} \frac{1}{(1-F(v))} \Big(\frac{d}{dv}(\theta^x(v)\ln\theta(v)) \Big) dv \geq \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)}{x}$$
$$\iff \int_{\underline{v}_r}^{\bar{v}_r} \frac{1}{(1-F(v))} \Big(\frac{d}{dv}(\theta^x(v)\ln\theta(v)) \Big) dv \geq \theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)$$
$$\iff \frac{\theta^x(v)\ln\theta(v)}{1-F(v)}\Bigg|_{\underline{v}_r}^{\bar{v}_r} - \int_{\underline{v}_r}^{\bar{v}_r} \theta^{1+x}(v)\ln\theta(v)dv \geq \theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)$$
$$\iff \frac{\theta^x(\bar{v}_r)\ln\theta(\bar{v}_r)F(\bar{v}_r)}{1-F(\bar{v}_r)} \geq \int_{\underline{v}_r}^{\bar{v}_r} \theta^{1+x}(v)\ln\theta(v) + \frac{\theta^x(\underline{v}_r)\ln\theta(\underline{v}_r)}{1-F(\underline{v}_r)}$$

By integration by parts on (3), we get

$$\int_{\underline{v}_r}^{\bar{v}_r} \frac{\theta'(v)}{(n-1)\theta^{\frac{n-2}{n-1}}(v)(1-F(v))} dv = \big(\theta(\bar{v}_r)\big)^{\frac{1}{n-1}}$$
$$\Rightarrow \int_{\underline{v}_r}^{\bar{v}_r} \frac{d\theta^{\frac{1}{n-1}}(v)}{dv} \frac{1}{(1-F(v))} dv = \big(\theta(\bar{v}_r)\big)^{\frac{1}{n-1}}$$
$$\Rightarrow \frac{\theta^{\frac{1}{n-1}}(v)}{1-F(v)}\Bigg|_{\underline{v}_r}^{\bar{v}_r} - \int_{\underline{v}_r}^{\bar{v}_r} \theta^{\frac{n}{n-1}}(v)dv = \big(\theta(\bar{v}_r)\big)^{\frac{1}{n-1}}. \qquad (9)$$

The inequality now follows by noting that $\theta(v)$ is increasing in v. □

Finally observe that as the center's reserve price r tends to 1, both parameters $\underline{v}_r$ and $\bar{v}_r$ also tend to 1 because equations (2,3) imply that for $r < 1$, $r < \underline{v}_r < \bar{v}_r < 1$.

3.3 Numerical Study of the Equilibria

We present a selection of some plots depicting the numerical values of the equilibria as characterized by the analytical results above. In our numerical study, we assume that the private values of buyers are drawn i.i.d. from a uniform distribution f over $[0, 1]$.

Figure 1 shows the value of minimum and maximum reserve prices by the intermediary, $\bar{v}_r$ and $\underline{v}_r$, as a function of the reserve price r of the center. Figure 2 depicts the distribution π of reserve prices of the intermediary as a function of the number n of intermediaries for a fixed reserve price $r = 0$ of the center. As it can be seen from this figure, the distribution π of reserve prices becomes more concentrated toward $\bar{v}_r$ as the number n of intermediaries increases. Figure 3 depicts the distribution π of reserve prices of the intermediary as a function of the reserve price r for $n = 2$.

From Figure 1, we note that both $\bar{v}_r$ and $\underline{v}_r$ increase with r, showing that the intermediary must generally raise her

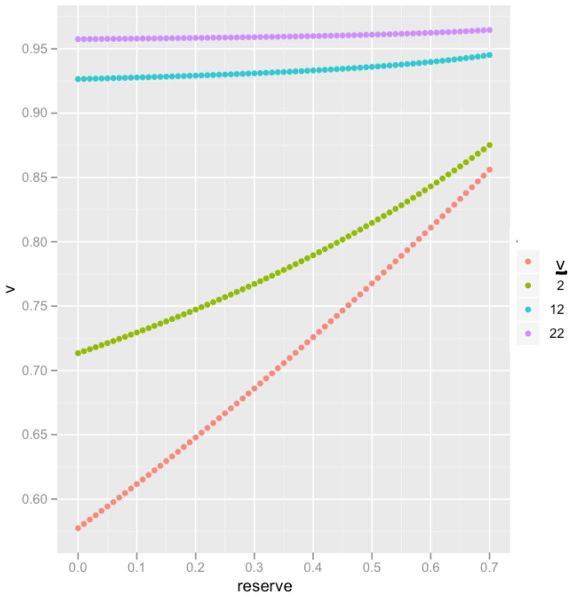

Figure 1: Interval parameters, $\bar{v}_r$ and $\underline{v}_r$, as a function of the reserve price r.

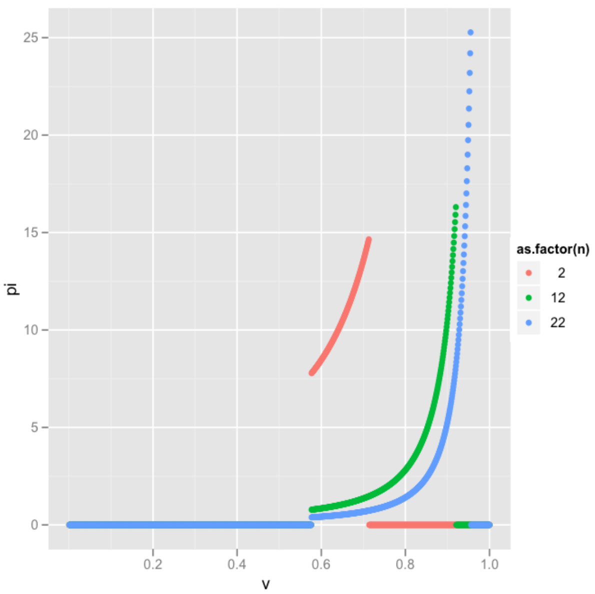

Figure 2: Probability distribution π as a function of the number of intermedieries n for a fixed reserve price $r = 0$.

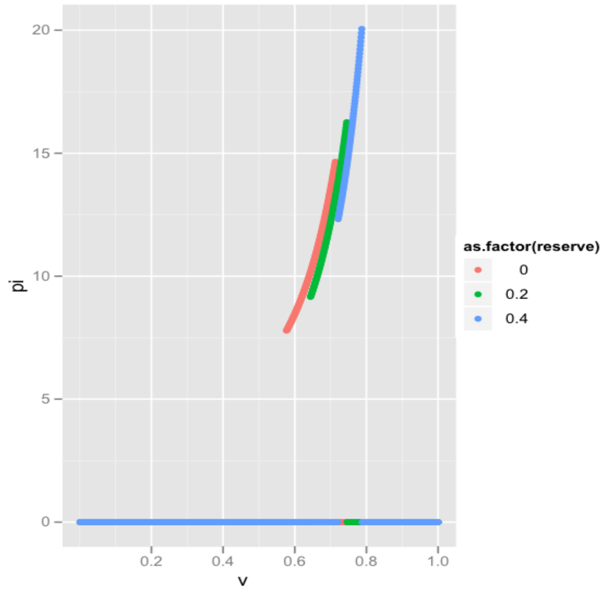

Figure 3: Probability distribution π as a function of changing reserve price r for a fixed $n = 2$.

prices as the center's reserve increases in order to take sufficient profit. We also see that only $\bar{v}_r$ increases with n; however Figure 2 illustrates the fact the the distribution mass shifts towards $\bar{v}_r$ as n increases. Similarly, for a fixed n the the distribution π also shifts upwards as r increases, as seen in Figure 3.

4. CENTER

Given that the center runs a second price auction with a reserve r, let G_r be the CDF of the implied distribution among intermediaries, i.e.:

$$G_r(v) = \begin{cases} \int_{\underline{v}_r}^{\bar{v}_r} \pi_r(v)F(v)dv & v \leq \underline{v}_r \\ \int_{\underline{v}_r}^{v} \pi_r(v)dv + \int_{v}^{\bar{v}_r} \pi_r(v)F(v)dv & v \in [\underline{v}_r, \bar{v}_r] \\ 1 & v \geq \bar{v}_r \end{cases}$$

with the implied density:

$$g_r(v) = \begin{cases} \pi_r(v)\big(1 - F(v)\big) & v \in [\underline{v}_r, \bar{v}_r] \\ 0 & \text{otherwise.} \end{cases}$$

Define the 'virtual value' of a valuation v as

$$\phi_r(v) = v - \frac{1 - G_r(v)}{g_r(v)}$$

Let the (equilibrium) probability of an intermediary who bids v winning the good be denoted $\alpha_r(v)$. Note that:

$$\alpha_r(v) = \begin{cases} 0 & v < r. \\ \alpha_r(\underline{v}_r) & v \in [r, \underline{v}_r) \\ \big(G_r(v)\big)^{n-1} & v \in [\underline{v}_r, \bar{v}_r] \\ 1 & v > \bar{v}_r. \end{cases}$$

Therefore, the problem of picking the optimal reserve price for the center can be written as:

$$\max_{r\in[0,1]} \int_r^1 \phi_r(v)g_r(v)\alpha_r(v)dv \qquad \text{(COPT)}$$
$$\text{s.t. } (2-4).$$

The main theorem of this section gives us a characterization of the center's problem. As a result of this approach, the objective function of the center can be written as an analytical function of exogenous parameters of the problem, and the two endogenous variables $\underline{v}_r$, $\bar{v}_r$, and therefore, compared to (COPT), where the objective function involves an integral containing the endogenously determined probability distribution π_r, this formulation is more amenable to numerical calculation.

THEOREM 4. *The optimal reserve prive for the center solves (where $\nu(v) = v - \frac{1-F(v)}{f(v)}$):*

$$\max_r \nu(\bar{v}_r) - r\left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}} - (n-1)c_r. \qquad \text{(COPT")}$$
s.t. $(2-4)$,

PROOF. Beginning with the r^* that solves COPT, we undertake some manipulation of the objective function.

$$\begin{aligned}
&\int_r^1 \phi_r(v)g_r(v)\alpha_r(v)dv \\
&= \int_r^{\underline{v}_r} \phi_r(v)g_r(v)\alpha_r(v)dv + \int_{\underline{v}_r}^{\bar{v}_r} \phi_r(v)g_r(v)\alpha_r(v)dv \\
&= -\int_r^{\underline{v}_r} \big(1-G_r(v)\big)\alpha_r(v)dv + \int_{\underline{v}_r}^{\bar{v}_r} \phi_r(v)g_r(v)\alpha_r(v)dv \\
&= -\big(1-G_r(\underline{v}_r)\big)\alpha_r(\underline{v}_r)\big(\underline{v}_r - r\big) \\
&\quad + \int_{\underline{v}_r}^{\bar{v}_r}\left(v - \frac{1-G_r(v)}{g_r(v)}\right)g_r(v)\alpha_r(v)dv \\
&= \underbrace{-\big(1-G_r(\underline{v}_r)\big)\alpha_r(\underline{v}_r)\big(\underline{v}_r - r\big)}_{(a)} + \underbrace{\int_{\underline{v}_r}^{\bar{v}_r} vg_r(v)\alpha_r(v)dv}_{(b)} \\
&\quad \underbrace{-\int_{\underline{v}_r}^{\bar{v}_r}\alpha_r(v)dv}_{(c)} + \underbrace{\int_{\underline{v}_r}^{\bar{v}_r} G_r(v)\alpha_r(v)dv}_{(d)}
\end{aligned}$$

Now, to further simplify each of the terms. First

$$\begin{aligned}
(a) &= \big(1-G_r(\underline{v}_r)\big)\alpha_r(\underline{v}_r)\big(\underline{v}_r - r\big) \\
&= \big(1-G_r(\underline{v}_r)\big)\alpha_r(\underline{v}_r)\frac{1-F(\underline{v}_r)}{f(\underline{v}_r)} \\
&= c_r\frac{1-G_r(\underline{v}_r)}{1-F(\underline{v}_r)}.
\end{aligned}$$

where the second line follows from (2) and the the third follows from (5).

Next,

$$(c) = \int_{\underline{v}_r}^{\bar{v}_r}\alpha_r(v)dv = c_r\int_{\underline{v}_r}^{\bar{v}_r}\theta(v)dv = \frac{c_r}{1-F(v)}\Bigg|_{\underline{v}_r}^{\bar{v}_r}.$$

Finally,

$$\begin{aligned}
(d) &= \int_{\bar{v}_r}^{\underline{v}_r} G_r(v)\alpha_r(v)dv \\
&= c_r^{\frac{n}{n-1}}\int_{\underline{v}_r}^{\bar{v}_r}\theta^{\frac{n}{n-1}}(v)dv \\
&= c_r^{\frac{n}{n-1}}\left(\frac{\theta^{\frac{1}{n-1}}(v)}{1-F(v)}\Bigg|_{\underline{v}_r}^{\bar{v}_r} - c_r^{-\frac{1}{n-1}}\right),
\end{aligned}$$

where the second equality follows from (4) and the third follows from (9). Therefore,

$$-(a) - (c) + (d) = -c_r,$$

and as a result COPT can be written as:

$$\max_r \int_{\underline{v}_r}^{\bar{v}_r} vg_r(v)\alpha_r(v)dv - c_r. \qquad \text{(COPT')}$$
$$\text{s.t. } (2-4).$$

Recall that for $v \in [\underline{v}_r, \bar{v}_r]$, $\alpha_r(v) = c_r\theta(v)$ and $g_r(v) = \pi_r(v)(1-F(v))$. Substituting $\pi_r(v)$ from (4),

$$\begin{aligned}
&\int_{\underline{v}_r}^{\bar{v}_r} vg_r(v)\alpha_r(v)dv \\
&= c_r^{\frac{n}{n-1}}\int_{\underline{v}_r}^{\bar{v}_r}\frac{v}{n-1}\theta'(v)\theta^{\frac{1}{n-1}}(v)dv \\
&= \frac{c_r^{\frac{n}{n-1}}}{n}\int_{\underline{v}_r}^{\bar{v}_r} v\frac{d}{dv}\theta^{\frac{n}{n-1}}(v)dv \\
&= \frac{c_r^{\frac{n}{n-1}}}{n}\left(v\theta^{\frac{n}{n-1}}(v)\Big|_{\underline{v}_r}^{\bar{v}_r} - \int_{\underline{v}_r}^{\bar{v}_r}\theta^{\frac{n}{n-1}}(v)dv\right) \\
&= \frac{c_r^{\frac{n}{n-1}}}{n}\left(v\theta^{\frac{n}{n-1}}(v)\Big|_{\underline{v}_r}^{\bar{v}_r} - \frac{\theta^{\frac{1}{n-1}}(v)}{1-F(v)}\Bigg|_{\underline{v}_r}^{\bar{v}_r} + c_r^{-\frac{1}{n-1}}\right) \\
&= \frac{c_r^{\frac{n}{n-1}}}{n}\left(\theta^{\frac{n}{n-1}}(v)\left(v - \frac{1}{(1-F(v))\theta(v)}\right)\Bigg|_{\underline{v}_r}^{\bar{v}_r} + c_r^{-\frac{1}{n-1}}\right) \\
&= \frac{c_r^{\frac{n}{n-1}}}{n}\left(\theta^{\frac{n}{n-1}}(v)\left(v - \frac{1-F(v)}{f(v)}\right)\Bigg|_{\underline{v}_r}^{\bar{v}_r} + c_r^{-\frac{1}{n-1}}\right) \\
&= \frac{1}{n}\left(v - \frac{1-F(v)}{f(v)}\right)\left(\frac{\theta(v)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}}\Bigg|_{\underline{v}_r}^{\bar{v}_r} + \frac{c_r}{n} \\
&= \frac{1}{n}\left(\nu(\bar{v}_r) - r\left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}} + c_r\right)
\end{aligned}$$

where the third equality follows from integration by parts, the fourth by substituting in (9), and the last by recalling (2). Therefore, COPT' can be rewritten as stated in the theorem. □

Using the characterization given in Theorem 4, one can compute the revenue-maximizing reserve price of the center in the equilibrium. In fact, by performing a numerical study of those reserve prices (which is shown in Figure 4), we observe the following properties of such optimal reserve prices:

- As the number of intermediaries increases, the revenue-maximizing reserve price decreases.

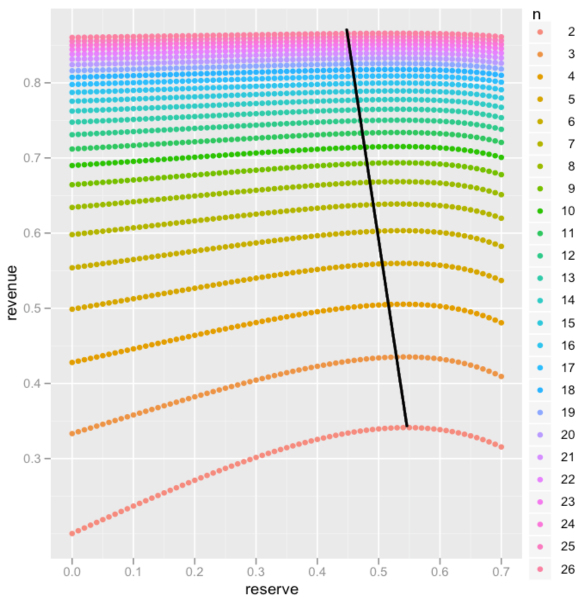

Figure 4: Center's revenue as a function of reserve price r and number of intermediaries n. The bold segment highlights the optimal reserve price achieving the maximum revenue for each n.

- As the number of intermediaries increases, the effect of the reserve price in maximizing the revenue decreases, i.e., as n increases, the difference in the revenue of the center as a function of its reserve price becomes smaller.

The above observations may imply that the choice of zero reserve price becomes optimal as the number of ad networks become large. On the contrary, we can analytically show that the center's optimal reserve price always remains strictly positive. The proof is in the Appendix.

THEOREM 5. *Let $r^\star$ be the center's optimal reserve price that solves COPT. Then $r^\star > 0$ for any number of intermediaries.*

5. MANY BUYERS PER INTERMEDIARY

In the previous sections, we analyzed our setting when each intermediary has exactly 1 buyer captive with him. We now let each intermediary have multiple buyers. First we look at the case where each intermediary has k buyers. Analogous to the case of a single buyer, we assume that each intermediary offers a second price auction with reserve. We note that there are dominant strategy incentive compatible mechanisms that are not second price auctions. To see this, note that, for example, a posted price mechanism is also dominant strategy incentive compatible. For the same reason as in the single buyer case, in equilibrium, each intermediary randomizes over reserve price. However, we are unable to offer the same explicit analytical characterization as (2 - 4). Still, our analysis below should be useful.

THEOREM 6. *Suppose there are $n \geq 2$ intermediaries, each of whom have exactly k advertisers. Each advertiser's valuation is an i.i.d. draw from a distribution f, F over [0,1] which satisfies the increasing hazard rate condition. Further suppose the center announces a second price auction with reserve r. There exists a symmetric equilibrium in mixed strategies among the intermediaries, where each intermediary offers prices $v \in [\underline{v}_r, \bar{v}_r]$ with density $\pi_r(v)$, where $\underline{v}_r, \bar{v}_r$ and $\pi(\cdot)$ jointly solve the following equations:*

1. *for all $\rho \in [\underline{v}_r, \bar{v}_r]$, $\frac{dP(\rho)}{d\rho} = 0$,*
2. *$P(\rho) = \int_\rho^1 k(k-1)(1-F(v))f(v)F^{k-2}(v)(\int_r^v \alpha(t)dt)dv + (\int_r^\rho \alpha(t)dt)k(1-F(\rho))F^{k-1}(\rho)$,*
3. *$\alpha(v) = 0$ for $v < r$, and $\alpha(v) = F_{\pi_r}^{n-1}(v)$ for $v \geq r$,*
4. *and finally,*

$$F_{\pi_r}(v) = \begin{cases} \int_{\underline{v}_r}^{\bar{v}_r} \pi_r(\rho)F^k(\rho)d\rho, \ \ \textit{for } v < \underline{v}_r \\ F_{\pi_r}(0) + \int_{\underline{v}_r}^{v} \pi_r(t)k(1-F(t))F^{k-1}(t) + \\ (\int_{\underline{v}_r}^{v} \pi_r(t)dt)k(k-1)(1-F(t))f(t)F^{k-2}(t)dt \\ \qquad \textit{for} \ \ v \in [\underline{v}_r, \bar{v}_r] \\ F_{\pi_r}(\bar{v}_r) + \int_{\bar{v}_r}^{v} k(k-1)(1-F(t))f(t)F^{k-2}(t)dt \\ \qquad \textit{for} \ \ v > \bar{v}_r \end{cases}$$

PROOF. First note that Lemma 1 still holds in this setting. As a result intermediaries must randomize over some interval $[\underline{v}_r, \bar{v}_r]$. Further, if an intermediary offers his k buyers a second price auction with reserve ρ; the second price in the intermediaries auction, which his valuation, is distributed as:

1. $v > \rho$ with density $k(k-1)(1-F(v))f(v)F^{k-2}(v)$.
2. $v = \rho$ with probability $k(1-F(\rho))F^{k-1}(\rho)$.
3. 0 with probability $F^k(\rho)$.

As in the case of the single buyer per intermediary, let $\alpha(v)$ be the probability the intermediary wins the good in the center's auction if he bids v.

Therefore if the seller runs a second price auction with revenue r, and bids v the intermediary's profit is $\int_r^v \alpha(t)dt$.

The intermediary's expected profit when he uses a second price auction with reserve ρ therefore is:

$$\int_\rho^1 k(k-1)(1-F(v))f(v)F^{k-2}(v)\left(\int_r^v \alpha(t)dt\right)dv \quad + \\ \left(\int_r^\rho \alpha(t)dt\right)k(1-F(\rho))F^{k-1}(\rho).$$

Suppose each intermediary randomizes over reserve price ρ with the distribution π_r over $[\underline{v}_r, \bar{v}_r]$. The generated distribution over second prices is:

1. $v > \bar{v}_r$ with density $k(k-1)(1-F(v))f(v)F^{k-2}(v)$.
2. $v \in [\underline{v}_r, \bar{v}_r]$ with density $\pi_r(v)k(1-F(v))F^{k-1}(v) + (\int_{\underline{v}_r}^v \pi_r(t)dt)\ k(k-1)(1-F(v))f(v)F^{k-2}(v)$.
3. 0 with probability $\int_{\underline{v}_r}^{\bar{v}_r} \pi_r(\rho)F^k(\rho)d\rho$.

Denote the resulting cumulative distribution as F_{π_r}. Note that α that results is:

$$\alpha(v) = \begin{cases} 0 & v < r \\ F_{\pi_r}^{n-1}(v) & v \geq r. \end{cases}$$

This concludes the proof. □

6. CONCLUSIONS

In this paper we have studied auction design in the presence of intermediaries. We have found that combining the notion of optimal auction design with the double-marginalization arising from the presence of intermediaries yields new strategic elements not present in either setting individually. In particular we proved that the optimal strategy of an intermediary setting a fixed reserve price is to choose this price non-deterministically over a particular interval. With these distributions endogenous, the center will then choose a reserve price that varies with the number of bidders, in contrast to a classical auction setting without intermediaries.

There are many possible directions for future work in this area. Most immediately, in the case of multiple buyers per intermediary, one would like to show that a second price auction with fixed reserve is the revenue-optimal incentive-compatible option available to the intermediaries, as we conjecture to be the case. Moreover, it would be interesting to see the effect of allowing buyers to choose their intermediary in the model. We suspect that this causes a fundamental change in strategy, along the lines of what occurs in the case of competing auctioneers [1, 13].

Finally, our model for auctions with intermediaries was inspired by the ad exchange systems, and it ignores many aspects of such auction systems [11]. Adopting auction theory to deal with these arising auction systems in presence of an interconnected network of intermediaries and buyers is an interesting research direction.

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APPENDIX

A. PROOFS

A.1 Proof of Proposition 1

- $a(\cdot)$ INCREASING: Incentive compatibility for a type v requires that for any $v' \neq v$:

$$va(v) - p(v) \geq va(v') - p(v'). \tag{10}$$

Fix $v, v' < v$. IC also requires that:

$$v'a(v') - p(v') \geq v'a(v) - p(v). \tag{11}$$

Adding the two inequalities, we have that:

$$\begin{aligned} va(v) + v'a(v') &\geq va(v') + v'a(v) \\ \Rightarrow (v - v')(a(v) - a(v')) &\geq 0. \end{aligned}$$

Since $v - v' > 0$, it follows that $a(v) \geq a(v')$.

- $p(\cdot)$ IS CHARACTERIZED BY (1): Assume that there exists an incentive compatible pricing rule $p(\cdot)$. Define

$$U(v) = va(v) - p(v).$$

The two incentive compatibility conditions for v and $v' < v$ can be written as:

$$\begin{aligned} U(v) &\geq va(v') - p(v'), \\ U(v') &\geq v'a(v) - p(v). \end{aligned}$$

Therefore we have that

$$(v - v')a(v') \leq U(v) - U(v') \leq (v - v')a(v).$$

Divide throughout by $v - v'$:

$$a(v') \leq \frac{U(v) - U(v')}{v - v'} \leq a(v).$$

As $v' \uparrow v$, we have:

$$\frac{dU(v)}{dv} = a(v),$$

and integrating:

$$U(v) = \int_0^v a(t)dt + c.$$

Therefore, since $U(v) = va(v) - p(v)$, we have that:

$$p(v) = va(v) - \int_0^v a(t)dt - c.$$

Finally, note that $p(\cdot) \geq 0$ by profit maximization, $U(0) = -p(0)$ by definition, and $U(0) \geq 0$ by individual rationality. Therefore $p(0) = 0$, and as a result we have that:

$$p(v) = va(v) - \int_0^v a(t)dt.$$

A.2 Proof of Theorem 5

PROOF. The objective function (COPT") can be rewritten as:

$$\underbrace{\bar{v}_r - \frac{1}{((1-F(\bar{v}_r))\theta(\bar{v}_r)}}_{(a)} \underbrace{- r\left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}}}_{(b)} \underbrace{- \frac{n-1}{\theta(\bar{v}_r)}}_{(c)}$$

Note that

$$\frac{d}{dr}(a) = \frac{d\bar{v}_r}{dr} - \frac{f(\bar{v}_r)}{(1-F(\bar{v}_r))^2\theta(\bar{v}_r)}\frac{d\bar{v}_r}{dr} + \frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)(1-F(\bar{v}_r))}\frac{d\bar{v}_r}{dr}$$
$$= \frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)(1-F(\bar{v}_r))}\frac{d\bar{v}_r}{dr}.$$
$$\frac{d}{dr}(c) = (n-1)\frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)}\frac{d\bar{v}_r}{dr}.$$
$$\Rightarrow \frac{d}{dr}((a)+(c)) = \frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)}\frac{d\bar{v}_r}{dr}\left(\left(\frac{1}{1-F(\bar{v}_r)}-1\right)+n\right)$$

Substituting in (7) we have:

$$\frac{d}{dr}((a)+(c)) = \underbrace{n\frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)}\frac{d\bar{v}_r}{dr}}_{(d)} + \underbrace{\frac{\theta'(\underline{v}_r)}{\theta^{\frac{n}{n-1}}(\bar{v}_r)\theta^{\frac{n-2}{n-1}}(\underline{v}_r)(1-F(\underline{v}_r))}\frac{d\underline{v}_r}{dr}}_{(e)} \tag{12}$$

Further,

$$\frac{d}{dr}(b) = \underbrace{-\left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}}}_{(f)} \underbrace{- r\frac{d}{dr}\left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}}}_{(g)} \tag{13}$$

$$\begin{aligned}(e)+(f) &= \frac{\theta'(\underline{v}_r)}{\theta^{\frac{n}{n-1}}(\bar{v}_r)\theta^{\frac{n-2}{n-1}}(\underline{v}_r)(1-F(\underline{v}_r))}\frac{d\underline{v}_r}{dr} - \left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}} \\ &= \left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}}\left(\frac{\theta'(\underline{v}_r)}{\theta^2(\underline{v}_r)(1-F(\underline{v}_r))}\frac{d\underline{v}_r}{dr} - 1\right) \\ &= 0,\end{aligned}$$

Where the third equation follows from noting that

$$\frac{\theta'(\underline{v}_r)}{\theta^2(\underline{v}_r)(1-F(\underline{v}_r))}\frac{d\underline{v}_r}{dr} = \frac{d}{dr}\nu(\underline{v}_r) = \frac{d}{dr}r = 1.$$

Therefore,

$$\frac{d}{dr}(a)+(b)+(c) = n\frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)}\frac{d\bar{v}_r}{dr} - r\frac{d}{dr}\left(\frac{\theta(\underline{v}_r)}{\theta(\bar{v}_r)}\right)^{\frac{n}{n-1}}. \tag{14}$$

Note that, by (14)

$$\frac{d}{dr}OBJ\Big|_{r=0} = n\frac{\theta'(\bar{v}_r)}{\theta^2(\bar{v}_r)}\frac{d\bar{v}_r}{dr}\Big|_{r=0}.$$

Note that

$$\begin{aligned}\theta'(v) &= \frac{d}{dr}\frac{f(v)}{(1-F(v))^2} \\ &= \frac{f'(v)(1-F(v))^2+2f^2(v)(1-F(v))}{(1-F(v)^4} \\ &= \frac{f'(v)(1-F(v))+2f^2(v)}{(1-F(v)^3}\end{aligned}$$

Recall that by the monotone hazard rate assumption, $\frac{f(v)}{1-F(v)}$ is increasing in v, i.e.

$$\begin{aligned}&\frac{d}{dv}\frac{f(v)}{1-F(v)} \geq 0 \\ \Rightarrow\ &\frac{f'(v)(1-F(v)+f^2(v)}{(1-F(v))^2} \geq 0 \\ \Rightarrow\ &\frac{f'(v)(1-F(v)+2f^2(v)}{(1-F(v))^2} \geq 0 \\ \Rightarrow\ &\frac{f'(v)(1-F(v)+2f^2(v)}{(1-F(v))^3} \geq 0 \\ \Rightarrow\ &\theta'(v) \geq 0\end{aligned}$$

By (7), we have

$$\frac{\theta'(\bar{v}_r)}{\theta^{\frac{n-2}{n-1}}(\bar{v}_r)}\frac{d\bar{v}_r}{dr}\left(\frac{1}{1-F(\bar{v}_r)}-1\right) = \frac{\theta'(\underline{v}_r)}{\theta^{\frac{n-2}{n-1}}(\underline{v}_r)(1-F(\underline{v}_r))}\frac{d\underline{v}_r}{dr}$$

Note that $\frac{d\underline{v}_r}{dr} \geq 0$ by Theorem 3. Therefore,

$$\frac{d}{dr}OBJ\Big|_{r=0} \geq 0,$$

concluding the proof. □